Rigorizing and Extending the Cox–Jaynes Derivation of Probability: Implications for Statistical Practice

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Summary. There have been three attempts to date to establish foundations for the discipline of probability, namely the efforts of Kolmogorov (who rigorized the frequentist approach), de Finetti (who gave Bayesian notions of belief and betting odds a formal treatment) and RT Cox/ET Jaynes (who developed a Bayesian theory of probability based on reasonable expectation (Cox) and the optimal processing of information (Jaynes)). The original “proof” of the validity of the Cox–Jaynes approach has been shown to be incomplete, and attempts to date to remedy this situation are themselves not entirely satisfactory. Here we offer a new axiomatization that both rigorizes the Cox–Jaynes derivation of probability and extends it — from apparent dependence on finite additivity to (1) countable additivity and (2) the ability to simultaneously make uncountably many probability assertions in a logically-internally-consistent manner — and we discuss the implications of this work for statistical methodology and applications. This topic is sharply relevant for statistical practice, because continuous expression of uncertainty — for example, taking the set Θ of possible values of an unknown θ to be (0, 1), or R, or the space of all cumulative distribution functions on R — is ubiquitous, but has not previously been rigorously supported under at least one popular Bayesian axiomatization of probability. The most important area of statistical methodology that our work has now justified from a Cox–Jaynes perspective is Bayesian non-parametric (BNP) inference, a topic of fundamental importance in applied statistics. We present two interesting foundational findings: (1) Kolmogorov’s probability function PK(A) of the single argument A is isomorphic to a version of the Cox–Jaynes two-argument probability map PCJ(A | B) in which Kolmogorov’s B has been hard-wired to coincide with his sample space Ω, and (2) most or all previous BNP work has actually been foundationally supported by a hybrid frequentist-Bayesian version of Kolmogorov’s probability function in which parameters are treated as random variables (an unacceptable move from the frequentist perspective); this previous BNP work is methodologically sound but is based on an awkward blend of frequentist and Bayesian ideas (whereas our Cox–Jaynes BNP is purely Bayesian, which has interpretational advantages).

Keywords: Associativity equation, axioms, belief, betting odds, Boolean algebra, conditional probability, Cox’s Theorem, countable additivity, de Finetti axioms, finite additivity, foundations of probability, foundations of statistics, frequentist, information, Kolmogorov axioms, logical internal consistency, machine learning, optimal information processing, philosophy of science, principles, “qualitative correspondence with common sense”, reasonable expectation, Russell’s paradox, sequential continuity, Stone’s Representation Theorem, Zermelo–Fraenkel set theory.

1. Introduction

The systematic study of probability

(a) began (e.g., David [1962]) in 1654 with an exchange of letters (de Fermat and Pascal [1654]) between Pascal and Fermat, which yielded a notion of probability based on symmetry and equipossible outcomes;

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(b) was expanded into the realm of conditional probability by Bayes (1764) and Laplace (1820); and

(c) opened a completely new (frequentist) chapter with the work of Cournot (1843) and Venn (1888).

However, it was not until efforts by Kolmogorov (1933: frequentist), de Finetti (1931, 1937: Bayesian, based on belief and betting odds) and RT Cox/ET Jaynes (Bayesian, based on reasonable expectation (Cox, 1946, 1961, 1978) and optimal information-processing (Jaynes, 2003) that attempts were made to establish rigorous foundations for the frequentist and Bayesian approaches to probability. This work is not yet complete for the Cox–Jaynes (CJ) Bayesian approach; in this paper, we offer a formulation of the CJ system that both rigorizes and extends it.

Specifically, we propose a new set of axioms that imply all of the standard axioms of probability, including Countable Additivity, which does not follow from the basic de Finetti Axioms and has not been shown to arise naturally from previous CJ attempts. Then we show that the Kolmogorov approach is a special case of the CJ system: under our approach, which requires the use of conditional probability as the fundamental primitive upon which probability theory is developed, all of Kolmogorov’s Axioms become Theorems.

Remark 1.1. In what follows, with one exception, we have chosen simple examples in which statistical implications arise from probabilistic foundations. It may be objected that these examples are too simple to provide useful guidance in complex settings; we disagree with this view. Many of the most important foundational issues already arise in the simplest of settings, and extra complexity generally tends to obscure those issues. It will become clear how complex examples may be built up from simple problems without requiring any changes to the theory described here.

An outline of the rest of the paper is as follows. In Section 2 we summarize the three leading probability systems (we do not examine the Pascal–Fermat approach, in which the concept of equiprobability in effect uses probability circularly to define probability). Section 3 presents our Definitions and Axioms, and in Section 4 we give our Theorems and Proofs. Section 5 concludes the paper with a discussion, including implications of our work for methodological and applied statistical practice; we focus in particular on the crucial topic of Bayesian non-parametric (BNP) inference (see, e.g., [Hjort et al. (2010)] about cumulative distribution functions (CDFs) and regression/classification functions.

2. A brief summary of the Kolmogorov, de Finetti and Cox–Jaynes approaches to defining probability

2.1. Kolmogorov

Kolmogorov (1933) rigorized the frequentist intuition behind repeatedly choosing a point (atom) \( \omega \) at random inside a Venn diagram, in which

(a) one may take (overlapping or non-overlapping) circles of varying radii to represent sets \((A, B, \ldots)\) of \( \omega \) values,

(b) the sample space \( \Omega \) of all possible atoms is represented by the entire Venn-diagram rectangle, and

(c) all atoms are “equally probable.”
His original intent was to define probability $P_K$ as a map, from the power set $2^\Omega$ of all subsets of $\Omega$ to the unit interval $[0, 1]$, which satisfied a small set of “reasonable” Axioms whose purpose was to force $P_K$ to behave “sensibly” from a frequentist (repeated-sampling) point of view.

When the number of atoms in $\Omega$ is finite, this approach works without difficulty: no (set-theoretic) pathologies can occur. But when $|\Omega|\geq\aleph_0$ — i.e., when the number of atoms in $\Omega$ is countably or uncountably infinite — Kolmogorov ran into measure-theoretic trouble: for example, the power set $2^\Omega$ when $\Omega = \mathbb{R}^k$ is “too strange” to permit problem-free assignment of probabilities to all of its members (an example of what can go wrong is the von Neumann paradox (von Neumann (1929)), in which (assuming the Axiom of Choice) one can (i) partition the unit square in $\mathbb{R}^2$ (with area 1) into a finite number of subsets, (ii) act upon those subsets with an affine area-preserving transformation, and (iii) end up with two unit squares, with a total area of 2).

Kolmogorov thus had to settle, when $|\Omega|\geq\aleph_0$, for constructing $P_K$ on a subset $\mathcal{F}$ of $2^\Omega$, and he wanted $\mathcal{F}$ to have two properties:

(a) nothing pathological can happen when defining $P_K$ on members of $\mathcal{F}$, and

(b) (for greatest generality) $\mathcal{F}$ is in some sense the “largest” subset of $2^\Omega$ for which (a) is true.

He chose to restrict attention to sets in $2^\Omega$ that are part of a $\sigma$-algebra $\mathcal{F}$ on $\Omega$ (i.e., a collection of subsets of $\Omega$ that includes $\Omega$ and is closed under countable union, intersection and complementation) set operations). This gave rise to his familiar concept of a probability triple $(\Omega, \mathcal{F}, P_K)$, in which he imposed the following Axioms on the function $P_K: \mathcal{F} \to \mathbb{R}$:

- (Normalization) $P_K(\Omega) = 1$;
- (Non-Negativity) For all sets $A_i \in \mathcal{F}$, $P_K(A_i) \geq 0$; and
- (Countable Additivity) For all countable collections of disjoint sets $A_i \in \mathcal{F}$, $P_K(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P_K(A_i)$.

A fourth “Axiom,” which actually follows from the three Axioms above as a Theorem (see, e.g., Çinlar (2011)), is sometimes (e.g., Jaynes (2003)) also included:

- (Continuity from Above) If sets $A_i \in \mathcal{F}$ are such that the sequence $A_1 \supseteq A_2 \supseteq \ldots$ tends to the empty set $\emptyset$, then $P_K(A_i) \to 0$.

A well-used example of Kolmogorov’s paradigm arises when $\Omega = \mathbb{R}$ and $\mathcal{F}$ is taken to be the Borel $\sigma$-algebra on $\mathbb{R}$, formed from complements and countable unions and intersections of intervals of the form $(a,b), [a,b), (a,b\] and [a,b] for $a$ and $b$ real with $-\infty \leq a < b \leq \infty$.

**Remark 2.1.** Kolmogorov (1933) did not give a convincing motivation for his Countable-Additivity Axiom; as noted by Schervish et al. (1984), he adopted countable additivity “for expedience.” We explore the role of Countable Additivity below.

**Remark 2.2.** Note that Kolmogorov’s original probability map $P_K$ is a function of one argument; in other words, he did not regard conditional probability as the primitive concept of uncertainty quantification to be axiomatized. As Jaynes (2003) puts it,

[Kolmogorov] finds [conditional probability] an awkward notion, really unwanted, and mentions it only reluctantly, as a seeming afterthought. Although [Kolmogorov’s 1933 book] has a section entitled “Bayes’ Theorem,” most of his followers ignore it.

This is in sharp contrast to the position taken by the de Finetti and CJ systems, in which all probabilities $P_{deF}(\cdot | \cdot)$ and $P_{CJ}(\cdot | \cdot)$ are conditional. In Section 5 we discuss the implications of this fundamental difference.
Remark 2.3. It may be objected that Kolmogorov’s system is pure mathematics and is therefore not especially wedded to the frequentist perspective, but this point of view flies in the face of Kolmogorov’s own pronouncements on the subject; for example,

The basis for the applicability of the results of the mathematical theory of probability to real “random phenomena” must depend on some form of the frequency concept of probability, the unavoidable nature of which has been established by von Mises in a spirited manner (Kolmogorov (1933)).

All Bayesian statisticians would find fault with the word “unavoidable” in this quote; see Section 5 for comment.

2.2. De Finetti

De Finetti had goals that differed completely from those of Kolomogorov: de Finetti (1931, 1937) wanted to create a theory of probability based on degrees of belief — in the truth of collections of (true/false) propositions (not sets of atoms, as with Kolmogorov) of unknown truth status — in such a way that the degrees of belief are conditioned on experience. This automatically renders de Finetti’s approach personal, if only because two different individuals may well have experiential paths differing in ways that are relevant to the apparent truth of the proposition(s) at issue. De Finetti chose to accomplish his task by invoking the idea of betting odds, and he further chose to concentrate on what is now called the operational subjective (OS) approach to statistical modeling, in which the fact that real-world measurements can never in practice attain more than a finite number of possible values is central to his theory. For a broad and deep investigation into the implications of the de Finetti-OS methodology and world view in statistics, see Lad (1996) and the many interesting publications cited therein.

In a standard setting for de Finetti’s system, You (Good (1950)) — a person wishing to reason sensibly in the face of uncertainty — are wagering against an opponent $O$ about the truth of a proposition $A$ whose truth status is unknown to both You and $O$. Your job is to set the price $P_{deF}(You)$ today of a promise to pay 1 (small) monetary unit (MU) if $A$ turns out to be true, and the truth about $A$ will be revealed by a referee (who never lies) tomorrow. $O$ will be permitted today either

(a) to buy Your promise at Your set price, or

(b) to require You to buy that same promise from $O$ at the same price.

It develops that, in this setting, if You set an absurd price — bigger than 1 MU or smaller than 0 MUs, as it turns out — $O$ can force You to lose money no matter whether $A$ is true or false, an embarrassing outcome that is referred to as $O$ having made Dutch book against You. De Finetti elaborated this situation into one in which You are required to set simultaneous prices about two propositions $A$ and $B$, and showed that, unless Your prices satisfy what he called the Coherence Axioms, $O$ can always make Dutch book against You. In this way he derived (most of) the standard rules of probability, which Kolmogorov took as Axioms, as Theorems arising from the de Finetti Coherence Axioms.

To set up these Axioms (see, e.g., Fishburn (1986)), de Finetti invoked the primitive concept of $\succ$ as a comparative operation between two propositions $A$ and $B$, conditional on background experience $E$ (which we suppress here for notational simplicity): $A \succ B$ means that proposition $A$ is more probable for You than proposition $B$. He then defines $\sim$ by the relation

$$A \sim B \text{ iff neither } A \succ B \text{ nor } B \succ A,$$

and $\succeq$ by the further relation

$$A \succeq B \text{ iff either } A \succ B \text{ or } A \sim B.$$
The de Finetti Coherence Axioms (acting on propositions \((A, B, C)\), regarded as elements of a universe of discourse \(\mathcal{U}_{deF}\), which includes \(\emptyset\) and \(T\), generic false and true propositions, respectively) are then as follows.

- **(Weak Order)** \(\succ\) is asymmetric (i.e., if \(A \succ B\) then not \((B \succ A)\)) and transitive (i.e., if \(A \succ B\) and \(B \succ C\) then \(A \succ C\)), and \(\sim\) is transitive;

- **(Non-Triviality)** \(\mathcal{U}_{deF} \succ \emptyset\);

- **(Non-Negativity)** For all \(A \in \mathcal{U}_{deF}\), \(A \succeq \emptyset\); and

- **(Additivity)** If \((A \text{ and } C) = (B \text{ and } C) = \emptyset\) then \(A \succ B\) is equivalent in truth value to \((A \text{ or } C) \succ (B \text{ or } C)\).

Taken together, these Axioms imply the existence of a function \(P_{deF} : (\mathcal{U}_{deF} \times \mathcal{U}_{deF}) \rightarrow [0, 1]\) that satisfies Kolmogorov’s Normalization and Non-Negativity Axioms. In this form, all that de Finetti obtains is Finite Additivity as a substitute for Kolmogorov’s Countable-Additivity Axiom.

### 2.3. Cox and Jaynes

Cox ([1946](#), [1961](#), [1978](#)) had an agenda that was even more basic — and in our view, more fundamental — than the program of study created by de Finetti. Motivated (see Snow (2001)) by ideas suggested by Schrodinger ([1947](#)), Cox began with a small set of natural-language Principles for how a plausibility function \(pl(A \mid B)\), acting on propositions \(B\) (regarded as true) and \(A\) (with unknown truth status), should behave, if this function is to accord with common sense and sound scientific thinking. He then distilled these Principles into a small set of Axioms, and (as with de Finetti) attempted to prove a Theorem showing that the usual basic rules of probability are implied by his Axioms. Jaynes ([2003](#)) sharpened and considerably extended Cox’s work; in the sketch of the CJ approach below, we adhere more closely to Jaynes than to Cox. (See Snow (2002) for a discussion of the fact that Cox’s publications in 1946, 1961 and 1978 actually developed three subtly different versions of Cox’s Theorem.)

Jaynes ([2003](#)) adopted the metaphor of constructing a robot that he wanted to program to embody the character (mentioned above) called You by Good ([1950](#)), namely an individual wishing to both reason and behave sensibly when uncertainty is present. Following but elaborating on Cox, Jaynes identified two Principles (he called them Desiderata) to which the robot must subscribe, as a starting point in the effort to construct a working version of You:

- **(Representation)** Degrees of plausibility \(pl(A \mid B)\) can be identified as unique real numbers \(p \in \mathbb{R}\); and

- **(Qualitative Correspondence With Common Sense; Logical Internal Consistency)** This has three sub-Principles:
  
  - If there is more than one path to a correct conclusion, all such paths must lead to the same plausibility result;
  
  - In assessing \(pl(A \mid B)\), You must always use all of the available information that You regard as relevant to the assessment; and
  
  - Equivalent states of information about \((A \mid B)\) always lead to the same \(pl(A \mid B)\).

From this, CJ distilled a set of three Axioms, as follows:

- **(Negation)** The plausibility of a proposition determines the plausibility of the proposition’s negation; each decreases as the other increases;
(Chain Rule for and) The plausibility of the conjunction \((AB)\) \(\triangleq (A \text{ and } B)\) of two propositions \((A, B)\) depends only on the plausibility of \(B\) and that of \(\{A \text{ given that } B \text{ is true}\}\) (or equivalently the plausibility of \(A\) and that of \(\{B \text{ given that } A \text{ is true}\}\)); and

(Consistency in Updating Information) Suppose that \((AB)\) is equivalent to \((CD)\); then if You acquire new information \(A\) and later acquire further new information \(B\), and correctly update all plausibilities each time, the updated plausibilities will be the same as if (knowing nothing about \(A\) and \(B\)) You had first acquired new information \(C\) and then acquired further new information \(D\).

Cox and Jaynes then attempted a proof (under several further technical assumptions) that all of the basic rules of probability (i.e., Kolmogorov’s Axioms and all correct Theorems that derive from those Axioms) follow as Theorems from the CJ Axioms, as instantiated through a function \(\mathbb{P}_{CJ}\) that maps an unspecified collection \(\mathcal{C} \times \mathcal{C}\) of propositions to \([0, 1]\). The basic idea has two parts, corresponding to the derivation of (i) the Product Rule, for probabilistically dealing with the logical relation \(\text{and}\) and (ii) the Sum Rule, for handling the special case of \(\text{or}\) that yields \(\mathbb{P}_{CJ}(A | B) + \mathbb{P}_{CJ}(\text{not } A | B) = 1\).

(Product Rule) By the Axiom governing the Chain Rule for \(\text{and}\), there exists a function \(F: \mathbb{R}^2 \to \mathbb{R}\) such that \(pl(AB | C) = F[pl(B | C), pl(A | BC)]\). Jaynes shows that the Axiom on Consistency in Updating Information forces \(F\) to satisfy the \(\text{Associativity Equation}\)

\[
F[F(x, y), z] = F[x, F(y, z)],
\]
whose solution can always be written in the form

\[
w(AB | C) = w(A | C) w(B | AC) = w(B | C) w(A | BC)
\]
for some positive continuous monotonic function \(w: \mathbb{R} \to \mathbb{R}\). Jaynes then shows that the convention \(0 \leq w(x) \leq 1\) may be adopted without loss of generality.

(Sum Rule) Under the Negation Axiom, there must exist a continuous monotonic-decreasing function \(S: [0, 1) \to [0, 1]\) such that \(w(\text{not } A | B) = S[w(A | B)]\), with \(S(0) = 1\) and \(S(1) = 0\). Jaynes shows that this, plus the CJ Consistency Axiom, forces \(S\) to satisfy the functional equation

\[
x S \left[ \frac{S(y)}{x} \right] = y S \left[ \frac{S(x)}{y} \right],
\]
in which \(x = w(A | C) > 0\) and \(y = w(B | C) > 0\). This now leads to the definition \(\mathbb{P}_{CJ}(x | 1) \triangleq w^m(x)\) for some positive real constant \(m\) (in which \(1\) is a generic true statement to be defined below), out of which the Product and (special-case, Finite-Additivity) Sum Rules become

\[
\mathbb{P}_{CJ}(AB | C) = \mathbb{P}_{CJ}(A | C) \mathbb{P}_{CJ}(B | AC) = \mathbb{P}_{CJ}(B | C) \mathbb{P}_{CJ}(A | BC)
\]
and

\[
\mathbb{P}_{CJ}(A | B) + \mathbb{P}_{CJ}(\text{not } A | B) = 1.
\]

As mentioned previously, neither Cox nor Jaynes showed that Countable Additivity follows from their assumptions; like de Finetti, they were fully content just with Finite Additivity. We return to this issue below.
2.4. Domain of the Kolmogorov, de Finetti and Cox–Jaynes probability functions; Stone’s Representation Theorem

Kolmogorov (1933) was clear about the domain of his probability function: having defined his sample space (set) $\Omega$, which in turn identifies his $\sigma$-algebra $\mathcal{F}$ of subsets of $\Omega$, $P_K$ is a function from $\mathcal{F}$ to $[0, 1]$. De Finetti and CJ, however, were less explicit on this point: $P_{deF}$ and $P_{CJ}$ both map from propositions to $[0, 1]$, but de Finetti never made limpidly clear the precise mathematical nature of his universe of discourse $\mathcal{U}_{deF}$, and (as noted above) in the writings of Cox and Jaynes the domain of $P_{CJ}$ is almost always simply a nearly unspecified collection ($\mathcal{E} \times \mathcal{E}$) of propositions. We say nearly because there is one quite restrictive form of $\mathcal{E}$–specification in the CJ system: Cox never showed any interest in an infinite $\mathcal{E}$, and Jaynes (2003) is vehement in what he refers to as his Finite-Sets Policy (pp. 43–44):

It is very important to note that our Consistency Theorems have been established only for probabilities assigned on finite sets of propositions. In principle, every problem must start with such finite-set probabilities; extension to [countably] infinite sets is permitted only when this is the result of a well-defined and well-behaved limiting process from a finite set.

As will become clear below, this position is overly cautious: we rigorize the CJ approach even when the number of propositions at issue is uncountable.

It would be nice if $\mathcal{E}$ could be equated to $\{\text{all true/false propositions}\}$, but to avoid Russell’s paradox (Russell (1903)) and other manifestations of unpleasantness, we subscribe to Zermelo-Fraenkel (ZFC) set theory (e.g., Fraenkel et al. (1958)). The domain of “all true/false propositions” is the largest possible Boolean algebra, and is thus a universal set, which does not exist in ZFC.

Some thought reveals that, to rigorize the CJ system without running afoul of paradoxes, $\mathcal{E}$ has to be based on a particular type of propositional Boolean algebra (see, e.g., Givant and Halmos (2009)): a six-tuple consisting of

- a finite or (countably or uncountably) infinite set $\mathcal{A}$ of (true/false) propositions $A, B, \ldots$,
- two binary operations and a unary operation on the propositions in $\mathcal{A}$: $\land$ (meet or and), $\lor$ (join or or), and $\neg$ (complement or not), and
- two special elements $0$ and $1$ (also called bottom and top, respectively),

such that

1. for all propositions $(A, B, C) \in \mathcal{A}$ the five Boolean Axioms in Table 3 (in the Appendix) hold, and moreover
2. $\mathcal{E}$ is closed under countable applications of $\land$ and $\lor$.

Sometimes it is convenient to use the term Boolean algebra just to refer to the set $\mathcal{A}$ of propositions in the six-tuple described above. We say that $\mathcal{E}$ has to be based on a Boolean algebra because — to conform to the CJ foundational Principle that all probabilities $P_{CJ}(A | B)$ are conditional — $P_{CJ}$ is actually a map from $[\mathcal{A} \times (\mathcal{A} \setminus 0)]$ to $[0, 1]$.

In his widely cited book, Jaynes (2003) claims that basing probabilities on propositions rather than on sets is more general (pp. 651–652):

The Kolmogorov system of probability is a game played on a sample space $\Omega$ of elementary propositions $\omega_i$. ... But a proposition $A$ referring to the real world cannot always be viewed as a disjunction of elementary propositions $\omega_i$ from any set $\Omega$ that has meaning in the context of our problem. The attempt to replace logical operations on
the propositions $A, B, \ldots$ by set operations on the set $\Omega$ does not change the abstract nature of [Kolmogorov’s] theory, but it makes it less general in respects that can matter in applications. Therefore we [Cox and Jaynes] have sought to formulate probability theory in the wider sense of an extension of Aristotelian logic.

In his many seminal publications on probability over a 50–year period, Jaynes got many things right, but this is a mathematical error. There is a fundamental Theorem due to Stone (1936), the informal statement of which, in the context of our paper, is as follows.

**Theorem 2.4 (Stone’s Representation Theorem).** Every Cox–Jaynes propositional Boolean algebra $(\mathcal{A}, \land, \lor, \neg, 0, 1)$ is isomorphic to a Kolmogorov set-theoretic space $(\mathcal{F}, \cap, \cup, \cdot^c, \emptyset, \Omega)$.

In the Appendix (Tables 4 and 5) we give two simple examples of this isomorphism, which permits us in what follows to

(a) formulate our Definitions, Axioms and Theorems set-theoretically, and

(b) interpret them propositionally.

### 2.5. State of play with the Cox–Jaynes system prior to this paper

One of the most astute commentators on the CJ system is Halpern (1999a,b), who is sympathetic to the CJ goals but who showed that the CJ “proof” is incomplete. Halpern did this by constructing an explicit counterexample, in which he exhibits a (somewhat strange) plausibility function $p(A | B)$ that (i) satisfies the CJ Axioms but (ii) violates the Product Rule. (Cox’s published work was correctly characterized by Snow (1998), who charitably describes Cox’s writing as exhibiting a “discursive, decidedly unsyllogistic, expository style.” Snow construes a sentence in Cox (1946) to mean that Cox did in fact avoid Halpern’s later counterexample, via additional subtly-stated assumptions; we, however, agree instead with Halpern that Cox’s lack of rigor left some important work undone.)

Halpern points out that Paris (1994), who is also sympathetic to CJ, found a way to salvage Cox’s Theorem that is immune to Halpern’s counterexample; however, Paris’ method involves an additional Axiom that has a strongly undesirable consequence, as follows. Taking advantage of Stone’s Representation Theorem to permit the domain of the CJ plausibility function $p(\cdot | \cdot)$ to be set-theoretic, Paris’ extra Axiom may be stated as follows:

- **(Density)** For all $0 \leq \alpha, \beta, \gamma \leq 1$ and $\epsilon > 0$, there exist sets $U_1 \supseteq U_2 \supseteq U_3 \supseteq U_4$ such that

  (a) $U_3 \neq \emptyset$, (b) $|p(U_4 | U_3) - \alpha| < \epsilon$, (c) $|p(U_3 | U_2) - \beta| < \epsilon$ and (d) $|p(U_2 | U_1) - \gamma| < \epsilon$.

This is referred to as a Density Axiom because it implies that, for any given $B$, the set of values taken on by $p(A | B)$ as $A$ varies is dense in $[0, 1]$. Halpern (1999a) points out that Paris’ Density Axiom forces the domain of the CJ probability function to be (at least countably) infinite, which Halpern correctly regards as an unacceptable restriction to the scope of the CJ approach. (Van Horn (2003) uses the primitive concept of state of information to motivate a Density Axiom, similar to Paris’, that Van Horn finds less objectionable, but the idea of “state of information” is defined at an intuitive, non-rigorous level in his paper, making it difficult to evaluate Van Horn’s approach formally.)

**Other previous work.** A number of other approaches to rigorizing CJ, including extensions of the methods previously mentioned, have also been proposed.

- Dupré and Tipler (2006) have suggested an alternative method based on retraction mappings (see, e.g., Hatcher (2002)), and have shown that with this motivation in mind, a
proof that combines aspects of Cox’s system and de Finetti’s approach can be written in a highly concise fashion \cite{Dupre2009}. Note that these authors take as an Axiom that probability satisfies strong rescaling, which is a powerful assumption: by the Riesz-Markov-Kakutani Representation Theorem (see, e.g., \cite{Halmos1978}), all bounded linear monotone functionals can be represented as integrals, so that assuming strong rescaling is equivalent to assuming that probability is an integral.

The main issue with this approach, however, is that it is restricted to settings in which the domain of the unknowns of inferential interest forms a partially ordered commutative algebra (such as $\mathbb{R}^k$); it therefore is not applicable to problems — e.g., involving graphical networks, trees and protein-folding structures — in which no such partial ordering exists (our formulation of CJ handles these settings with no difficulty).

- \textbf{Hardy (2002)} offers another alternative axiomatization, based on assumptions of linear order, dividedness, and Archimedianity.

- \textbf{Knuth and Skilling (2012)} propose a variation on and extension of Cox’s proof using lattice symmetries, but their derivation assumes that the domain of $P_{CJ}$ is finite, and this does not map readily to settings (such as BNP modeling: see Section 5) in which the truth status of an uncountably infinite number of propositions is at issue.

- \textbf{Arnborg and Sjodin (1999)} offer an alternative to Cox’s proof with a refinability Axiom that replaces Paris’ Density Axiom, with the intent of ruling out Halpern’s counterexample. Their subsequent work \cite{Arnborg2003} also lists other possible Axioms to replace Density, along with a discussion regarding the implications of both Cox’s proof and de Finetti’s proof.

- \textbf{Zimmerman and Cremers (2011)} propose another variation on a proof of Cox’s Theorem, and make a number of remarks regarding the implications of Cox’s proof for the process of quantifying uncertainty.

- \textbf{Colyvan (2004, 2008)} has pointed out that Cox’s Theorem can fail to hold when the underlying logic of reasoning is not Boolean.

All of this work is important and useful, but none of these authors have proposed a system that has been shown to satisfy Countable Additivity; thus they are following paths that are distinct from the one we are attempting here.

\textbf{Remark 2.6 (Finite/infinite domains; Finite/Countable Additivity).} It is evident from a study of the CJ literature across multiple subject-matter areas (the dominant disciplines are statistics, philosophy and physics) that anyone wishing to build a probability function from first principles must make two separate binary choices, both involving the word “finite”:

- (Finiteness of Domain) Should the domain of Your probability map $P$ be finite or (countably or uncountably) infinite?
• (Finiteness of Additivity) Should Your probability function \( P \) be Finitely or Countably Additive?

Interaction exists between these two distinct choices; for example, it does not make sense to simultaneously choose (Finite Domain, Countable Additivity). It appears that there are three leading joint choices across these two dimensions, as follows:

(A) [Finite Sets] Only finite sets are allowed when axiomatizing probability theory (e.g., Jaynes);

(B) [Finite Additivity] Infinite sets are allowed, but You can only reason about them using finite additivity (e.g., de Finetti);

(C) [Countable Additivity] Infinite sets are allowed, without any cardinality restrictions, and You can reason about them using Countable Additivity (e.g., this paper).

In our approach, we take position (C), which in our view is of greatest interest for statistical modeling, because it allows us to define distributions on uncountable sets such as \( \mathbb{R} \), or (as in BNP) the space of all CDFs on \( \mathbb{R} \).

We have no universal quarrel with Finitely Additive probability calculations on domains consisting of finite sets; we acknowledge the existence of some settings — some motivated by physics and studied by some of the authors mentioned above, others illustrated in, e.g., [Lad 1996] — in which taking position (A) and performing a finite-set analysis may correctly and reasonably be of primary interest. On the other hand, we do not believe position (B) to be a practical approach for contemporary statistical modeling: for example, BNP is a non-starter when working solely with Finite Additivity.

To summarize the state of play with the CJ system to date,

(a) Cox and Jaynes achieved (i) compelling motivation for, and (ii) elegance of, their probability function \( P_{CJ} \), but not (iii) rigor and (iv) success in dealing with infinity;

(b) Halpern provided an explicit counterexample to the CJ “proof” of the validity of \( P_{CJ} \);

(c) Paris offered a new Axiom that defeats Halpern’s counterexample, but the resulting system does not permit the domain of \( P_{CJ} \) to be finite; and

(d) Many authors have tried to circumvent Paris’ Axiom, but none of their approaches has been demonstrated to achieve Countable Additivity.

In this paper we offer a new axiomatization that rigorizes \( P_{CJ} \) in a way that copes with finite or (countably or uncountably) infinite Boolean domains and is based on Countable Additivity.

3. Definitions and Axioms

**Remark 3.1.** By Stone’s Representation Theorem, we are free to rigorize Cox–Jaynes either in the context of propositional Boolean algebras or set-theoretic spaces; for convenience we choose the latter in our Definitions, Axioms and Theorems, and we then discuss the implications of our results for the propositional approach in Section 5.

**Remark 3.2.** In building the CJ probability system, our strategy is to construct a sequence \( P^{(1)} \)–\( P^{(4)} \) of functions, each obeying more of the basic rules of probability than the previous one, until the CJ probability function can finally be recognized as \( P^{(4)} \).

**Definition 3.3 (Domain of Probability Function).** Let \( \Omega \) be a finite or (countably or uncountably) infinite set, and let \( \mathcal{F} \) be a \( \sigma \)-algebra on \( \Omega \), closed (as usual) under countable union, intersection and complementation operations.
Remark 3.4. Here and throughout, we adopt the notational convention that any sets (e.g., $A, B, A_1, A_2, C, U, V, X, Y, D$) mentioned are — unless otherwise stated — elements of $\mathcal{F}$.

**Axiom 3.5 (Probability is a Real Number).** Let $\mathbb{P}^{(1)} : \mathcal{F} \times (\mathcal{F} \setminus \emptyset) \to R \subseteq \mathbb{R}$ be a function, written using the notation $\mathbb{P}^{(1)}(A \mid B)$. (Here $R$ is a closed subset of $\mathbb{R}$ with other structure that will become apparent in the Proofs in Section 4.)

**Remark 3.6.** Following Jaynes (2003) and many other writers, the essential assumption here is that, for each unique choice of $[A \in \mathcal{F}, (B \neq \emptyset) \in \mathcal{F}]$, $\mathbb{P}^{(1)}(A \mid B)$ is a unique (singleton) real value. Other (interval rather than singleton) theories of probability — e.g., belief functions (Dempster 1967), Shafer (1976) — are possible; we do not pursue them here.

**Axiom 3.7 (Sequential Continuity).** For non-empty $B$,

(a) Let $A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots$ such that $A_i \searrow A$. Then $\mathbb{P}^{(1)}(A_i \mid B) \searrow \mathbb{P}^{(1)}(A \mid B)$.

(b) Let $A_1 \supseteq A_2 \supseteq A_3 \supseteq \ldots$ such that $A_i \nearrow A$. Then $\mathbb{P}^{(1)}(A_i \mid B) \nearrow \mathbb{P}^{(1)}(A \mid B)$.

**Justification.** This Axiom is just a technical way to state the following natural criterion, in CJ language: given a (true) proposition $B$, if a proposition $A$ becomes (upon receipt of new information) more true (more false) by an infinitesimal amount, the function $\mathbb{P}^{(1)}(A \mid B)$ can increase (decrease) by at most an infinitesimal amount. This is similar to Kolmogorov’s “Axiom 4,” and is vital in what follows: it makes possible our proofs of Proposition 4.3 (the Associativity Equation), Theorem 4.7 (Normalization), and Theorem 4.11 (Countable Additivity).

Note that this Axiom’s negation implies that there can exist two events $A$ and $A^*$ that can be made arbitrarily close (in the set-theoretic sense of distance used in the Axiom) and yet $\mathbb{P}(A \mid B)$ and $\mathbb{P}(A^* \mid B)$ do not converge to the same value for any non-empty $B$. We find compelling the idea that as two events become indistinguishable, their probabilities must also become indistinguishable; hence this Axiom.

**Remark 3.8.** Continuity is the reason for our interest in Countable, rather than Finite, Additivity: one of the by-products of our results below is the following Theorem, stated informally:

**Theorem** ($(FA + C) \iff CA$) (Your probability function is Finitely Additive and satisfies Continuity) if and only if (Your probability function is Countably Additive).

This result (see Theorem 4.11 below for a discussion) has the following consequences:

- You wish to quantify Your uncertainty about a proposition $A$ with a Finitely Additive probability map $\mathbb{P}(\cdot \mid \cdot)$.

- You assume the negation of Countably Additivity; since You previously assumed Finite Additivity, this implies the negation of Continuity.

- You notice that You can find propositions $A_1, A_2, \ldots$ such that $\lim_{n \to \infty} A_n = A$; in other words, as information about the truth of $A$ increases, the information states ($A_i$ is true) get closer to, and eventually coincide with, the information state ($A$ is true).

- But there exists a $B$ such that $\mathbb{P}(\lim_{n \to \infty} A_n \mid B) \neq \mathbb{P}(A \mid B)$, since Continuity does not hold! Furthermore, the negation of Continuity implies that there exists such a pair of events $(A, B)$ in every non-Countably-Additive space. Note also, from Bayes’ Theorem, that this problem also occurs in calculations of the form $\mathbb{P}(B \mid A_i)$.

- If You think of $A_n$ as a data-learning process with $A$ being the event whose truth status You want to know, the negation of Countable Additivity implies that there exists a proposition that You cannot learn about as a sequence, even in the limit. Furthermore, $A$ is arbitrary: if someone asks “Why do You want to learn about $A$?”, the answer is, “Why not?”. 

Rigorizing and Extending Cox–Jaynes
Axiom 3.9 (Comparability: Product Rule). $P^{(1)}(AB \mid C)$ can be written as

$$f\left[ P^{(1)}(U \mid V), P^{(1)}(X \mid Y) \right]$$

for some function $f: (R \times R) \to R$ and some sets $(U, V, X, Y) \in \mathcal{F}$ involving $(A, B, C)$.

Axiom 3.10 (Comparability: Sum Rule). There exists a function $h: R \to R$ such that $P^{(1)}(A^c \mid B) = h\left[ P^{(1)}(A \mid B) \right]$.

Remark 3.11. Axioms 3.9 and 3.10 correspond to the CJ Axioms for the Chain Rule for \{and\} and for Negation, respectively. See Jaynes (2003) for a justification of these assumptions, which we — and many others (e.g., Tribus (1969), Fine (1973)) — find compelling.

Axiom 3.12 (Comparative Extendability). Either the range of $P^{(1)}$ is dense in $R$, or (if not) the following is true:

- Define $S \triangleq [\min\{R\}, \max\{R\}]$, $\Omega_E \triangleq \Omega \times S$ and $\mathcal{F}_E \triangleq \mathcal{F} \otimes \mathcal{B}(S)$, where $\mathcal{B}(S)$ is the Borel $\sigma$-algebra on $S$ and $\otimes$ in this context specifies a $\sigma$-algebra on the product space of $\mathcal{F}$ with $\mathcal{B}(S)$.

Remark 3.13. By Axiom 3.7 (Sequential Continuity), the range of $R$ is dominated by $P^{(1)}(\Omega \mid \Omega)$, ensuring the existence of $\max\{R\}$; a similar argument guarantees the existence of $\min\{R\}$, and thus of $S$.

- Then we can define an extended probability function

$$P^{(E)}: \mathcal{F}_E \times (\mathcal{F}_E \setminus \emptyset) \to S$$

with the following properties:

- $P^{(E)}$ satisfies all previous axioms, with Product-Rule function $f$ and Sum-Rule function $h$ identical to those in the construction of $P^{(1)}$;
- For any pairs $(A, X), (B, Y)$ such that $(A, B) \in \mathcal{F}$ and $(X, Y) \in \mathcal{B}(S)$ with $(B, Y) \neq \emptyset$, we have

$$P^{(E)}[(A, X) \mid (B, Y)] = f\left[ P^{(1)}(A \mid B), P^{(U)}(X \mid Y) \right].$$

Here $P^{(U)}(X \mid Y) \triangleq j[\text{Leb}(X \cap Y), \text{Leb}(Y)]$ — in which $\text{Leb}(\cdot)$ is Lebesgue measure on $R$ — satisfies all previous axioms with identical functions $f$ and $h$, and with some function $j: (R \times R) \to R$ normalized so that $P^{(U)}(S \mid S) = P^{(1)}(\Omega \mid \Omega)$ and $P^{(U)}(\emptyset \mid S) = P^{(1)}(\emptyset \mid \Omega)$.

Justification. This Axiom is needed to defeat Halpern’s (1999a) counterexample without appealing to Paris’ (1994) Density Axiom, which (as noted above) yields the unacceptable consequence that the domain of the CJ probability function must be infinite. The axiom appears complicated but actually instantiates a simple idea, namely that the triple $(\Omega, \mathcal{F}, P^{(1)})$ You start with must be reasonable enough to allow You, if You choose to do so, to

(a) extend the range of $P^{(1)}$ to an interval and

(b) bolt on an unrelated uniform random draw (uniform in the Lebesgue sense, not necessarily in the $P^{(1)}$ sense).
This is an example of a standard move in foundational work of this type; as Halpern (1999b) points out, Savage (1954) and Snow (1998) make similar moves. The intuition is that if You are prepared to evaluate probabilities on a finite number of atoms, You should be equally prepared to generate an unrelated uniform variate and ignore it. This is essentially always true when reasoning about uncertainty in the context of a real-world problem: it is inconceivable to ask a question so degenerate that it cannot possibly be considered, if the choice were made to do so, at the same time as an irrelevant uniform random number.

The above Axiom implies the following:

(1) By Sequential Continuity and properties of Lebesgue measure, $j$ is continuous, increasing in its first argument, and decreasing in its second argument;

(2) $j$ is essentially a warping/twisting/scaling function that tells You what it means to be uniform with respect to $P^{(1)}$ and its product rule, and this Axiom is equivalent to stating that there exists a uniform distribution, with respect to a measure that can be extended into two arguments, such that it has the same product rule as $P^{(1)}$;

(3) Bolting on the uniform must preserve the product rule of the original triple, which is the key property that rules out the Halpern counterexample; and

(4) You can easily recover $P^{(1)}(E \mid F) = P^{(E)}[(E, S) \mid (F, S)]$.

Remark 3.14. The subsequent proof proceeds by first establishing the Cox–Jaynes Theorem for triples where the range of $P^{(1)}$ is dense in R, and then proving it for everything else.

Remark 3.15. From a purely mathematical perspective, our Comparative Extendability Axiom is at least as strong as Paris' Density Axiom, and any resulting strengthening of the CJ assumptions in this way is not our intention. From a philosophical and applied-statistics perspective, however, we argue that Comparative Extendability is much more natural than Density, and this is our motivation. We are open to the possibility of a more elegant solution that rules out Halpern’s counterexample in spaces where it applies; such a solution has not yet been found, and is the subject of active research. In any case, while possibly not necessary, our argument is sufficient for our purposes.

4. Theorems and Proofs

Lemma 4.1 (Consistency). Let $(A, B, C, D) \in \mathcal{F}$ and notice that therefore $(AB, BC) \in \mathcal{F}$. Then $P^{(1)}(ABC \mid D) = P^{(1)}((AB)C \mid D) = P^{(1)}(A(BC) \mid D)$.

Proof. This follows immediately from the definitions of and (propositions) and intersection (sets).

Lemma 4.2 (Domain of Product Rule).

$$P^{(1)}(AB \mid C) = f \left[ P^{(1)}(B \mid C), P^{(1)}(A \mid BC) \right]$$

$$= f \left[ P^{(1)}(A \mid C), P^{(1)}(B \mid AC) \right],$$

in which $f$ is the function defined in Axiom 3.9.

Proof. See Tribus (1969), who showed by exhaustive enumeration — in the context of a propositional Boolean-algebra domain — that the two equalities in (11) are the only possibilities that satisfy logical internal consistency.
**Proposition 4.3 (Associativity Equation).** \( f(x, y) \) as defined in Axiom 3.9 satisfies 

\[
f(x, y) = g^{-1}[g(x) + g(y)]
\]

for a continuous strictly increasing function \( g: R \rightarrow R \).

**Proof.** Let \((A, B, C, D) \in \mathcal{F}\) and consider \(P^{(1)}(A B C | D)\). By Lemma 4.1 (Consistency), this is equal to both \(P^{(1)}((A B)C | D)\) and \(P^{(1)}(A(B C) | D)\). Then, using the function \( f \) in Lemma 4.2,

\[
P^{(1)}(A B C | D) = f \left[ P^{(1)}(B C | D), P^{(1)}(A | B C D) \right]
\]

and

\[
P^{(1)}(A B C | D) = f \left[ P^{(1)}(C | D), P^{(1)}(B | B C D) \right],
\]

Hence \( f \) must satisfy the functional equation \( f[f(x, y), z] = f[x, f(y, z)] \). There are now two cases to consider.

**Case 1:** Suppose that the range of \(P^{(1)}\) is dense in \(R \subset R\). See Aczél (1966) and Craigen and Páles (1989), who show that the solution of the functional equation in \( f \) is \( f(x, y) = g^{-1}[g(x) + g(y)] \) for some continuous strictly increasing function \( g: R \rightarrow R \).

**Case 2:** Now suppose instead that the range of \(P^{(1)}\) is not dense in \(R \subset R\). Then, by Axiom 3.12 (Comparative Extendability), extend \(P^{(1)}\) to \(P^{(E)}\). Since \( \mathcal{F}_E \) is a \(\sigma\)-algebra, Lemmas 4.1 and 4.2 and the previous portion of this proof apply to \(P^{(E)}\), just as they do to \(P^{(1)}\). But by Axiom 3.7 (Sequential Continuity), the range of \(P^{(E)}\) is dense in \(R\), and we get the solution in Case 1 for \( f \) for the full domain of \(P^{(E)}\). But \(P^{(1)}(E | F) = P^{(E)}((E, S) | (F, S))\) on its entire domain, and thus — since the solution for \( f \) applies to \(P^{(1)}\) as well — we arrive at the same result as in Case 1.

**Remark 4.4.** We have assumed that \(\mathcal{F}\) contains at least four elements. This is not a practical restriction, however, because the only \(\sigma\)-algebra with fewer than four elements is the degenerate \(\sigma\)-algebra \(\{\emptyset, \Omega\}\), for which the CJ Theorem is uninteresting.

**Remark 4.5 (Some Solutions of the Associativity Equation).** Notice that \( f(x, y) = x y \) satisfies the Associativity Equation: in our final version of probability, this is of course the solution that will be used. Notice further that \( f(x, y) = 100 x y \) also satisfies the Associativity Equation. In a non-traditionally-normalized theory of probability where \(P(\Omega | A) = 100\), this is the solution that would be used. It will be shown (Theorem 4.7) that no generality is lost by restricting the range of probability functions to \([0, 1]\).

**Theorem 4.6 (Product Rule).** There exists a \(P^{(2)}: \mathcal{F} \times (\mathcal{F} \setminus \emptyset) \rightarrow R\) such that \(P^{(2)}(A B | C) = P^{(2)}(A | B C) P^{(2)}(B | C) = P^{(2)}(B | A C) P^{(2)}(A | C)\).

**Proof.** By the Domain of the Product Rule (Lemma 4.2) and Proposition 4.3, we know that

\[
g \left[ P^{(1)}(A B | C) \right] = g \left[ P^{(1)}(B | C) \right] + g \left[ P^{(1)}(A | B C) \right]
\]

and

\[
g \left[ P^{(1)}(A | C) \right] + g \left[ P^{(1)}(B | A C) \right].
\]
Then
\[
\exp \left\{ g \left[ \mathbb{P}^{(1)}(A B \mid C) \right] \right\} = \exp \left\{ g \left[ \mathbb{P}^{(1)}(B \mid C) \right] \cdot \exp \left\{ g \left[ \mathbb{P}^{(1)}(A \mid B C) \right] \right\} \right. \\
= \exp \left\{ g \left[ \mathbb{P}^{(1)}(A \mid C) \right] \cdot \exp \left\{ g \left[ \mathbb{P}^{(1)}(B \mid A C) \right] \right\} \right. . \tag{16}
\]

Let \( \mathbb{P}^{(2)}(A \mid B) \triangleq \exp \left\{ g \left[ \mathbb{P}^{(1)}(A \mid B) \right] \right\} \). Clearly \( \mathbb{P}^{(2)} \) satisfies the Product Rule.

**Theorem 4.7 (Normalization).** There exists a \( \mathbb{P}^{(3)} : \mathcal{F} \times (\mathcal{F} \setminus \emptyset) \to \mathbb{R} \) such that (for any non-empty \( B \in \mathcal{F} \)) \( \mathbb{P}^{(3)}(\Omega \mid B) = 1 \), \( \mathbb{P}^{(3)}(\emptyset \mid B) = 0 \), and \( 0 \leq \mathbb{P}^{(3)}(A \mid B) \leq 1 \) for all \( A \in \mathcal{F} \).

**Proof.** From Axiom 3.7 (Sequential Continuity), we know that \( \mathbb{P}^{(1)} \) is bounded by \( \mathbb{P}^{(1)}(\Omega \mid B) \) for any non-empty \( B \in \mathcal{F} \). Notice that \( A \subseteq \Omega \) and \( \emptyset \subseteq A \) for any \( A \in \mathcal{F} \), and thus by Sequential Continuity (for any non-empty \( A \in \mathcal{F} \)) it must be true that \( \min \{ \mathbb{P}^{(1)} \} = \mathbb{P}^{(1)}(\emptyset \mid A) \) and \( \max \{ \mathbb{P}^{(1)} \} = \mathbb{P}^{(1)}(\Omega \mid A) \). Also note that \( g \) is strictly increasing and continuous on the full domain of \( \mathbb{P}^{(1)} \), and is thus bounded. Similarly, \( \exp(x) \) is continuous, strictly increasing and bounded on all domains of the form \([a, b] \) with \( a, b \neq \pm \infty \). Thus \( \mathbb{P}^{(2)} \) is bounded with \( \min \{ \mathbb{P}^{(2)} \} = \mathbb{P}^{(2)}(\emptyset \mid A) > -\infty \) and \( \max \{ \mathbb{P}^{(2)} \} = \mathbb{P}^{(2)}(\Omega \mid A) < \infty \) for any non-empty \( A \). For any non-empty \( B \), let
\[
\mathbb{P}^{(3)}(A \mid B) \triangleq \frac{\mathbb{P}^{(2)}(A \mid B) - \min \{ \mathbb{P}^{(2)} \}}{\max \{ \mathbb{P}^{(2)} \} - \min \{ \mathbb{P}^{(2)} \} } = \frac{\mathbb{P}^{(2)}(A \mid B) - \mathbb{P}^{(2)}(\emptyset \mid B)}{\mathbb{P}^{(2)}(\Omega \mid B) - \mathbb{P}^{(2)}(\emptyset \mid B)}. \tag{17}
\]

Clearly \( \mathbb{P}^{(3)} \) satisfies Normalization.

**Proposition 4.8 (Self-Reciprocal Function).** There exists a function \( k : \mathbb{R} \to \mathbb{R} \) such that \( \mathbb{P}^{(3)}(A^c \mid B) = k \left[ \mathbb{P}^{(3)}(A \mid B) \right] \) for all \( [A, (B \neq \emptyset)] \in \mathcal{F} \), with functional form \( k(x) = (1 - x^m)^{1/m} \) for some \( m > 0 \).

**Proof.** Let \( B \in \mathcal{F} \) be non-empty. By Axiom 3.10 (on the Existence of the Sum Rule), we know that there exists an \( h : \mathbb{R} \to \mathbb{R} \) such that \( \mathbb{P}^{(1)} (A^c \mid B) = h \left[ \mathbb{P}^{(1)}(A \mid B) \right] \). But since \( g(x), \exp(x) \) and the rescaling function used in constructing \( \mathbb{P}^{(3)} \) are all strictly increasing, it follows that they are 1–1, and that their composition, denoted \( l(x) \), is also 1–1 and hence invertible. Then define \( k(x) \triangleq l \left\{ h \left[ l^{-1}(x) \right] \right\} \), from which
\[
k \left[ \mathbb{P}^{(3)}(A \mid B) \right] = l \left\{ h \left( l^{-1} \left[ \mathbb{P}^{(3)}(A \mid B) \right] \right) \right\} = l \left\{ h \left[ \mathbb{P}^{(1)}(A \mid B) \right] \right\} \\
= l \left[ \mathbb{P}^{(1)}(A^c \mid B) \right] = \mathbb{P}^{(3)}(A^c \mid B); \tag{18}
\]
thus \( k \) exists for \( \mathbb{P}^{(3)} \). To find the functional form of \( k \), consider first the following.

The main line of the argument below involves forming fractions in which \( \mathbb{P}^{(3)}(A \mid C) \) and \( \mathbb{P}^{(3)}(B \mid C) \) serve as denominators (for arbitrary \( C \in \mathcal{F} \)), and of course we do not wish to divide by 0. So suppose temporarily that \( \mathbb{P}^{(3)}(A \mid C) = 0 \). From equation (18), \( k \left[ \mathbb{P}^{(3)}(A \mid C) \right] = \mathbb{P}^{(3)}(A^c \mid C) \). By temporary assumption, \( \mathbb{P}^{(3)}(A \mid C) = 0 = \mathbb{P}^{(3)}(\emptyset \mid C) = k \left[ \mathbb{P}^{(3)}(\Omega \mid B) \right] = k(1) \).

It follows by analogous argument that \( k(0) = 1 \), and it also follows by symmetry that this holds for \( \mathbb{P}^{(3)}(B \mid C) \), and hence without loss of generality we can assume that \( \mathbb{P}^{(3)}(A \mid C) \) and \( \mathbb{P}^{(3)}(B \mid C) \) are nonzero. The results \{\( k(0) = 1, k(1) = 0 \)\} will be taken henceforth as constraints (boundary conditions) for \( k \).
From the Product Rule, we know that
\[
\mathbb{P}^{(3)}(AB \mid C) = \mathbb{P}^{(3)}(A \mid C) \mathbb{P}^{(3)}(B \mid AC) = \mathbb{P}^{(3)}(B \mid C) \mathbb{P}^{(3)}(A \mid BC).
\]
(19)

By existence of \(k\),
\[
\mathbb{P}^{(3)}(A \mid C) k \left( \frac{\mathbb{P}^{(3)}(B^c \mid AC)}{\mathbb{P}^{(3)}(A \mid C)} \right) = \mathbb{P}^{(3)}(B \mid C) k \left( \frac{\mathbb{P}^{(3)}(A^c \mid BC)}{\mathbb{P}^{(3)}(B \mid C)} \right)
\]
(20)

But we also know that \(\mathbb{P}^{(3)}(AB^c \mid C) = \mathbb{P}^{(3)}(A \mid C) \mathbb{P}^{(3)}(B^c \mid AC)\), from which \(\mathbb{P}^{(3)}(B^c \mid AC) = \frac{\mathbb{P}^{(3)}(AB^c \mid C)}{\mathbb{P}^{(3)}(A \mid C)}\) for any \(A\) such that \(\mathbb{P}^{(3)}(A \mid C) > 0\). And similarly, \(\mathbb{P}^{(3)}(A^c B \mid C) = \mathbb{P}^{(3)}(B \mid C) \mathbb{P}^{(3)}(A^c \mid BC)\), which implies that
\[
\mathbb{P}^{(3)}(A^c \mid BC) = \frac{\mathbb{P}^{(3)}(A^c B \mid C)}{\mathbb{P}^{(3)}(B \mid C)} \quad \text{with } B \text{ such that } \mathbb{P}^{(3)}(B \mid C) > 0.
\]
(21)

Substituting this into the previous equation, it follows that
\[
\mathbb{P}^{(3)}(A \mid C) k \left( \frac{\mathbb{P}^{(3)}(AB \mid C)}{\mathbb{P}^{(3)}(A \mid C)} \right) = \mathbb{P}^{(3)}(B \mid C) k \left( \frac{\mathbb{P}^{(3)}(A^c B \mid C)}{\mathbb{P}^{(3)}(B \mid C)} \right)
\]
(22)

and this must hold for all non-empty \((A, B, C)\) with \(\mathbb{P}^{(3)}(A \mid C) > 0\) and \(\mathbb{P}^{(3)}(B \mid C) > 0\).

Consider \(B^c = AD = A \cap D\), where \(D \neq (A, B, C)\). Then \(A \cap B^c = A \cap (A \cap D) = A \cap D = B^c\) and \(B \cap A^c = (A \cap D)^c \cap A^c = (A^c \cup D^c) \cap A^c = A^c\). Substituting this into the previous equation, the result is \(\mathbb{P}^{(3)}(A \mid C) k \left[ \frac{\mathbb{P}^{(3)}(B^c \mid C)}{\mathbb{P}^{(3)}(A \mid C)} \right] = \mathbb{P}^{(3)}(B \mid C) k \left[ \frac{\mathbb{P}^{(3)}(A^c \mid C)}{\mathbb{P}^{(3)}(B^c \mid C)} \right] ; \) so \(k\) must satisfy the functional equation \(x \cdot \frac{k(y)}{x} = y \cdot \frac{k(x)}{y}\) subject to \(k(0) = 1\). Aczél (1966) has shown that this functional equation has the solution \(k(x) = (1 - x^m)^{1/m}\) for some constant \(m > 0\). Thus \(\mathbb{P}^{(3)}(A^c \mid B)^m = 1 - \mathbb{P}^{(3)}(A \mid B)^m\) for arbitrary positive constant \(m\).

**Theorem 4.9 (Sum Rule).** There exists a \(\mathbb{P}^{(4)} : \mathcal{F} \times (\mathcal{F} \setminus \varnothing) \rightarrow R\) such that \(\mathbb{P}^{(4)}(A^c \mid B) = 1 - \mathbb{P}^{(4)}(A \mid B)\) for all \([A, (B \neq \varnothing)] \in \mathcal{F}\).

**Proof.** From Proposition 4.8, we know that \(\mathbb{P}^{(3)}(A^c \mid B)^m = 1 - \mathbb{P}^{(3)}(A \mid B)^m\) for arbitrary positive constant \(m\) and for any \([A, (B \neq \varnothing)] \in \mathcal{F}\). Let \(\mathbb{P}^{(4)}(A \mid B) \triangleq \mathbb{P}^{(3)}(A \mid B)^m\). Clearly \(\mathbb{P}^{(4)}(A \mid B)\) both satisfies the Sum Rule and preserves the Product Rule.

**Proposition 4.10 (Finite Additivity).** For disjoint \(A_i \in \mathcal{F}\), \(\mathbb{P}^{(4)}\) satisfies
\[
\mathbb{P}^{(4)} \left( \bigcup_{i=1}^{n} A_i \mid B \right) = \sum_{i=1}^{n} \mathbb{P}^{(4)}(A_i \mid B)
\]
(23)

for any positive finite integer \(n\) and any non-empty \(B \in \mathcal{F}\).

**Proof.** This follows from Theorem 4.9 by induction.

**Theorem 4.11 (Countable Additivity).** For disjoint \(A_i \in \mathcal{F}\), \(\mathbb{P}^{(4)}\) satisfies
\[
\mathbb{P}^{(4)} \left( \bigcup_{i=1}^{\infty} A_i \mid B \right) = \sum_{i=1}^{\infty} \mathbb{P}^{(4)}(A_i \mid B)
\]
(24)

for any non-empty \(B \in \mathcal{F}\).
Furthermore, Kolmogorov’s axioms imply the axioms used to construct \( P \) for any non-empty set in \( \mathcal{F} \); define \( A_n^* = \bigcup_{i=1}^{n} A_i \subseteq \Omega \) and \( A = \bigcup_{i=1}^{\infty} A_i \subseteq \Omega \). Clearly \( A_n^* \subseteq A_{n+1}^* \) for all positive integers \( n \), with \( A_n^* \nsubseteq A \), so
\[
\mathbb{P}^{(4)}(\bigcup_{i=1}^{\infty} A_i | B) = \lim_{n \to \infty} \mathbb{P}^{(4)}(A_n^* | B) = \mathbb{P}^{(4)}(A | B). \tag{25}
\]

On the other hand, by Proposition 4.10 (Finite Additivity) and Axiom 3.7 (Sequential Continuity),
\[
\sum_{i=1}^{\infty} \mathbb{P}^{(4)}(A_i | B) = \lim_{n \to \infty} \mathbb{P}^{(4)}\left(\bigcup_{i=1}^{n} A_i | B\right) = \lim_{n \to \infty} \mathbb{P}^{(4)}(A_n^* | B) = \mathbb{P}^{(4)}(A | B). \tag{26}
\]

Furthermore, since \( A_n^* \nsubseteq A \), it follows that \( \mathbb{P}^{(4)}(A_n^* | B) \nsubseteq \mathbb{P}^{(4)}(A | B) \), from which \( \mathbb{P}^{(4)}\left(\bigcup_{i=1}^{\infty} A_i | B\right) = \mathbb{P}^{(4)}(A | B) \), as desired.

**Remark 4.12.** With reference to Remark 3.8, Theorem 4.11 has shown that \((FA+C) \rightarrow CA;\) as for the converse, \( CA \rightarrow FA \) is trivial, and \( CA \rightarrow C \) may be demonstrated with the same method as in Theorem 4.11.

**Remark 4.13.** At this point the Cox–Jaynes construction is complete: \( \mathbb{P}_{CJ} = \mathbb{P}^{(4)} \).

**Theorem 4.14 (Isomorphism).** There exists a function \( \mathbb{P}_{K(CJ)} : \mathcal{F} \to [0, 1] \), derived from the CJ system, such that all of Kolmogorov’s axioms hold for \( \mathbb{P}_{K(CJ)} \):

1. (Normalization) \( \mathbb{P}_{K(CJ)}(\Omega) = 1 \);
2. (Non-Negativity) \( \mathbb{P}_{K(CJ)}(A) \geq 0 \) for all \( A \in \mathcal{F} \); and
3. (Countable Additivity) For all countable collections of disjoint sets \( A_i \in \mathcal{F} \),
   \[
   \mathbb{P}_{K(CJ)}(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mathbb{P}_{K(CJ)}(A_i).
   \]

Furthermore, Kolmogorov’s axioms imply the axioms used to construct \( \mathbb{P}^{(1)} \) (and by proxy \( \mathbb{P}_{CJ} \)).

**Proof.** For any \( A \in \mathcal{F} \), let \( \mathbb{P}_{K(CJ)}(A) \triangleq \mathbb{P}^{(4)}(A | \Omega) \). Since statements (1), (2), and (3) in the Theorem hold for \( \mathbb{P}^{(4)} \), they must also hold for \( \mathbb{P}_{K(CJ)} \). Now define \( \mathbb{P}_{K(CJ)}(A | B) \triangleq \frac{\mathbb{P}_{K(CJ)}(A B)}{\mathbb{P}_{K(CJ)}(B)} \)
for any \( B \in \mathcal{F} \) with \( \mathbb{P}_{K(CJ)}(B) > 0 \) (see [Cinlar (2011)] for a full Kolmogorov treatment of the case \( \mathbb{P}_{K(CJ)}(B) = 0 \)). The resulting conditional version of \( \mathbb{P}_{K(CJ)} \) clearly satisfies all the Kolmogorov Axioms.

**Remark 4.15.** Thus Cox–Jaynes may be regarded as equivalent to Kolmogorov, but in a world in which conditional probability is the primitive concept.

5. Discussion

**Remark 5.1.** Why does this matter for the discipline of statistics?

- (Rigorization) Cox’s three seminal works (1946, 1961, 1978) have together been cited more than 1,900 times, and Jaynes’ book (2003) has been referenced almost 4,200 times; many people are using the CJ probability system without knowing that it has been, until now, foundationally flawed (in terms of both Halpern’s countereample and issues arising from the lack of explicitness regarding CJ’s domain).
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(Extension) No one in the Cox–Jaynes world to date — and this world is nontrivial in size — has been fully comfortable in dealing with countably and uncountably infinite numbers of propositions in their domains; this paper shows that this cautionary position is unnecessary. Indeed, the work presented here justifies for the first time a theory, based on the CJ system, of BNP inference (acting on uncountable numbers of propositions). We offer an applied example of CJ–BNP below.

Remark 5.2. What are the implications of this work for statistical methodology and applications?

(Conditional probability in the Kolmogorov approach) There is no longer any need for Kolmogorov-system practitioners to treat conditional probability as “an awkward notion, really unwanted” (Jaynes (2003)); our Theorem 4.14 shows that Kolmogorov probability \( P_K(\cdot) \) is equivalent to a version of the Cox–Jaynes system in which \( \Omega \) has been hard-wired into the second of the two inputs to \( P_{CJ}(\cdot|\cdot) \).

(There is no \( \mathbb{P} \)-uncertainty) In applied statistics, it is important to specify the uncertainty associated with the answers to questions \( Q \) that are framed with the help of a statistical model. In this context, Your total uncertainty \( \mathcal{U} \) can be partitioned into

- \( \theta \)-uncertainty, often called parameter uncertainty: the uncertainty contained inside a parametric statistical model, conditional on its stochastic assumptions (e.g., parametric families of distributions);
- \( \mathcal{M} \)-uncertainty, often called model uncertainty: the uncertainty about the choice of a statistical model among many that might be used to answer \( Q \); and
- \( \mathbb{P} \)-uncertainty, sometimes called systemic uncertainty: the uncertainty about the proper system of reasoning to use in constructing Your uncertainty quantification.

It is natural to ask how much uncertainty is associated with each of these three sources, and a variety of methods have been developed for answering such questions (see, e.g., the work by Leamer (1978); Draper (1993); Hoeting et al. (1999) and others on Bayesian model averaging).

The Cox–Jaynes (CJ) Theorem provides an extremely powerful result: if its assumptions (namely: probabilities are real numbers; sequential continuity; comparability for \( \text{and} \) and \( \text{not} \); and comparative extendability) are satisfied — and they are simple and general enough to be true a priori for virtually all real-world problems — then

(There is no \( \mathbb{P} \)-uncertainty) if and only if (the system of reasoning used is probability).

All uncertainty in probabilistic models is therefore the result of model specification: either via the parameters (\( \theta \)-uncertainty), or via the structure of the model itself (\( \mathcal{M} \)-uncertainty). This consequence illustrates the deep relationship between the CJ Theorem and applied statistics.

This result is also of fundamental importance for the field of machine learning: it follows that any learning algorithm that is not designed using probability theory possesses a non-zero level of \( \mathbb{P} \)-uncertainty.

Example 5.3. You are a machine learning researcher interested in a supervised-learning problem that can be solved via probability theory, but in a way that has been shown to be
Table 1. Descriptive summaries of a monetary outcome $y$ measured in two A/B tests $E_1$ and $E_2$ at eCommerce company X. SD = standard deviation; MU is explained in the text.

<table>
<thead>
<tr>
<th>Experiment</th>
<th>n</th>
<th>% 0</th>
<th>Mean</th>
<th>SD</th>
<th>Skewness</th>
<th>Kurtosis</th>
<th>Noise-To-Signal Ratio (SD/Mean)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_1$: T</td>
<td>12,234,293</td>
<td>90.7</td>
<td>9.128</td>
<td>129.7</td>
<td>157.6</td>
<td>59,247</td>
<td>14.21</td>
</tr>
<tr>
<td>$E_1$: C</td>
<td>12,231,500</td>
<td>90.7</td>
<td>9.203</td>
<td>147.8</td>
<td>328.9</td>
<td>266,640</td>
<td>16.06</td>
</tr>
<tr>
<td>$E_2$: T</td>
<td>128,349</td>
<td>70.1</td>
<td>1,080.8</td>
<td>33,095.8</td>
<td>205.9</td>
<td>52,888</td>
<td>30.62</td>
</tr>
<tr>
<td>$E_2$: C</td>
<td>128,372</td>
<td>70.0</td>
<td>1,016.2</td>
<td>36,484.9</td>
<td>289.1</td>
<td>92,750</td>
<td>35.90</td>
</tr>
</tbody>
</table>

The CJ Theorem demonstrates that there is always a penalty that is paid by all non-probabilistic algorithms in terms of uncertainty. The speed increase associated with moving to a polynomial-time non-probabilistic algorithm comes with the price of increased uncertainty associated with not having used probability. Crucially, this increase in uncertainty cannot be assessed by the model itself (unlike $\theta$–uncertainty), and cannot even be assessed by examining the model within a larger class of structurally similar models (unlike $M$–uncertainty). Some other method must be developed to assess the $P$–uncertainty that arises, when machine-learning algorithms are employed that have not been demonstrated to be instances of probability-based inferential, predictive or decision-theoretic methodologies.

- **(The value of Bayesian non-parametric modeling)** The above partitioning of total uncertainty $U$ into $(\theta, M, P)$ components also has connections with Bayesian non-parametrics. In our view, BNP is especially important in contemporary stochastic modeling, because de Finetti’s Representation Theorem for continuous real-valued outcomes (de Finetti (1937)) has the following informal content:

  - If Your uncertainty about real-valued observables \((y_1, y_2, \ldots)\) is exchangeable (see, e.g., Lindley and Novick (1981); Draper et al. (1993)), then — letting $F$ stand for the empirical CDF of \((y_1, y_2, \ldots)\), one possible logically-internally-consistent representation of Your predictive distribution for the vector \((y_1, \ldots, y_n)\) of the first $n$ observations, before You have seen them, is via the hierarchical model

    \[
    F \sim p(F) \\
    (y_i \mid F) \overset{IID}{\sim} F \quad (27)
    \]

    for \((i = 1, \ldots n)\), in which $p(F)$ is a probability distribution on the set $\mathcal{F}$ of all CDFs on $\mathbb{R}$.

    In other words, BNP priors on $\mathcal{F}$ arise directly from exchangeability judgments, which are at the heart of practical applied statistical work. As noted by many authors (e.g., Muller and Mitra (2013); Krnjajić et al. (2008)), BNP is a promising approach to a general strategy for coping with $M$–uncertainty.

**Remark 5.4.** The following case study is somewhat long and detailed; we offer it to provide an instance of BNP modeling that is completely justified by the CJ axioms. We address the foundational issues raised by this example in a Remark that immediately follows the case study.

**Example 5.5.** To illustrate the use of CJ–BNP modeling in a data-science context, consider the following analysis, in which one of us (DD) participated in 2014. eCommerce company X (identity suppressed for business confidentiality) routinely performs randomized controlled trials
(referred to in data science as A/B testing), in which users of X’s web page are assigned at random to a control group C (which receives the current best web experience) or to a treatment group T (which receives a modified version of C’s web experience in which a new intervention — such as enlarging photographs of products for sale — is incorporated). One of the principal outcome variables of interest is based on aggregate sales volume: let $y^T_i$ be the total gross value of all merchandise bought by $T$–group user i over a k-week period, with k typically on the order of 2–6 weeks, and let $y^C_j$ be the corresponding outcome for C–group user j in the same time period. For reasons of business confidentiality, all outcome values in this case study are expressed in monetary units (MU), which have been derived from US$ via a confidential monotone increasing transformation.

Table 1 offers descriptive summaries of the outcome $y$ in two experiments — code-named here $E_1$ and $E_2$ — at company X. $E_1$ was vastly larger than $E_2$: the total sample sizes in the two A/B tests were about 24,4 million and 257,000, respectively. On the recorded MU scale, about 90% of the outcome values in $E_1$ were 0 in both groups, with 70% the corresponding rate of 0s in $E_2$. The mean of $y$ in $E_1$ was about 9 MUs (9.1 T, 9.2 C), versus about 1,000 MUs (1,081 T, 1,016 C) in $E_2$; these experiments addressed sharply different sectors of the eCommerce business. Note that the distributions of $y$ in both T and C in both experiments are spectacularly long-right-hand-tailed, with skewness values ranging from about 160 to 330 and kurtosis values from about 55,000 to 270,000; the resulting noise-to-signal ratios (SD/mean) vary from 14 to 36, making it extremely difficult to find small (but still business-relevant) treatment differences even with sample sizes in the tens of millions.

To perform an inferential analysis of these experiments, imagine that one of them ($E_1$, say) has been run for a much longer period of time than it actually was, and assume stationarity of the effect of (T versus C) on $y$ during that period. Then it is sensible to speak of $\mu_T$ as the mean $y$ value You would observe in this longer period of time, if all users in the larger experiment had counterfactually received the treatment, and similarly for $\mu_C$. The unknown quantity of principal interest in eCommerce A/B tests is the relative difference between $\mu_C$ and $\mu_T$, referred to as the lift $\theta \triangleq \frac{\mu_T - \mu_C}{\mu_C}$. You could attempt a parametric (mixture-model) inferential analysis of $\theta$, but this would involve assumptions that do not arise directly from the context of the problem. Instead we proceed with a CJ–BNP analysis, as follows (see, e.g., Muller and Mitra (2013) for details on many of the steps described below).

Consider the observed values $y^T_i$ in the treatment group in one of the experiments. With the background information available to You, Your uncertainty about the $y^T_i$ is exchangeable, and by de Finetti’s Representation Theorem for real-valued outcomes You may use model (27), taking the $y^T_i$ to be conditionally IID from the underlying treatment CDF $F_T$ and specifying a prior for $F_T$. We choose to put a Dirichlet-process (DP) prior directly on $F_T$ (another alternative would be Dirichlet-process mixture modeling, in which each observation is allowed to have its own latent random effect and You put a DP on the latent-variables distribution, but this is highly computationally infeasible with 12 million observations in each of the T and C groups). Thus our model in the treatment group is

$$F_T \sim DP(\alpha_T, F_{0T})$$

and similarly (and independently, in parallel) for the C group; here $\alpha_T$ acts like a prior sample size and $F_{0T}$ is Your estimate of $F_T$ based on Your information external to the data.

By standard DP conjugate updating, the posterior on $F_T$ is another DP; with $y^T$ as the vector of all $n_T$ treatment $y$ values,

$$(F_T | y^T) \sim DP \left( \alpha_T + n_T, \frac{\alpha_T F_{0T} + \alpha_T n_T F^T_{\text{n}}} {\alpha_T + n_T} \right),$$

(29)

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in which \( \hat{F}_n^T \) is the empirical CDF based on \( y^T \). We now let \( \alpha_T \downarrow 0 \), to express the reality that little is known about \( F_T \) external to the experimental data vector \( y^T \); the resulting posterior is independent of \( F_0T \) and has the simple form

\[
(F_T \mid y^T) \sim DP(n_T, \hat{F}_n^T). \tag{30}
\]

\( F_T \) draws from the posterior in (30) may now be accomplished via the usual stick-breaking approach (Sethuraman (1994)), but Draper (2015) has established the following Fact, which offers an accurate and computationally considerably faster alternative.

**Fact (Draper (2015)).** For even rather small values of \( n \), the ordinary frequentist non-parametric bootstrap of Efron (1979) provides draws from \( DP(n, \hat{F}_n) \) that are

(a) indistinguishable from stick-breaking realizations and

(b) about 30 times faster (in clock time) to obtain.

Moreover, the frequentist bootstrap is embarrassingly parallel (see, e.g., Matloff (2011)); this is not true of standard instantiations of stick-breaking, in which the size of the next stick fragment depends sequentially on how much of the stick is left. We use the above **Fact** in what follows.

Figure 1 presents 100 posterior CDF draws for each of the \( T \) and \( C \) groups in each of \( E_1 \) and \( E_2 \), together with the posterior means of the CDF distributions. The CDFs in question are so unusual that care is required to plot them: the horizontal scale must be in \( \log(MU) \) and the vertical scale in \( \logit(F) \), with the large dot summarizing the point at which the distribution transitions from point-mass at 0 to positive values. All horizontal and vertical scales have been held constant across the four plots; this makes them directly comparable in a number of interesting ways, some of which are summarized in a different form in Table 1 but some of which are not:

- The basic shape of the log-logit mean curves is remarkably similar across the four combinations \((T, C) \times (E_1, E_2)\), revealing an aspect of stability in the underlying CDFs for this outcome variable \( y \) that had not previously been noticed; this can perhaps serve as the basis of future hierarchical (borrowing-strength) analyses involving other experiments.

- In experiment \( E_1 \), with 12 million observations in each group, posterior uncertainty about \( F \) does not begin to exhibit itself (reading left to right) until about \( e^9 \approx 8,100 \) MUs, which corresponds to the \( \logit^{-1}(0.99995) \) percentile; the corresponding figures in \( E_2 \), with sample sizes that are smaller by a factor of almost 100, are about 60,000 MUs and the 99.9th percentile. Of course, with the mean at stake and violently skewed and kurtotic distributions, very high percentiles are precisely the distributional locations of greatest leverage.

The \( F \) draws in Figure 1 induce BNP posterior distributions on \( \theta \) in both experiments; these are explored in Figure 2 and summarized in Table 2. The BNP posterior densities are plotted with solid curves, and large-sample Gaussian approximations are given in dotted lines; by Taylor expansion the posterior mean is approximately \( \bar{\theta} = \bar{y}_T - \bar{y}_C \), and the posterior SD is

\[
SD(\theta \mid y^T, y^C) \approx \sqrt{\frac{s_T^2}{\bar{y}_T^2 n_T} + \frac{s_C^2}{\bar{y}_C^2 n_C}}, \tag{31}
\]

in which the vector \((n_T, \bar{y}_T, s_T)\) records the sample size, mean and SD in the \( T \) group (and similarly for \( C \)). The Gaussian approximations are good, at least in the tails, in both of the full \( E_1 \) and \( E_2 \) experiments (indicated in the top row of Figure 2). The lower right plot in Figure 2 displays the results for a small subgroup (known in eCommerce as a segment) of users...
Fig. 1. 100 CJ–BNP posterior CDF draws (dotted lines) for each of the $C$ (column 1) and $T$ (column 2) groups in each of $E_1$ (row 1) and $E_2$ (row 2); solid curves plot the posterior means. The large dots in the lower left parts of each graph represent the points at which the distributions move from point-mass at 0 to positive values.

in $E_2$, identified by a particular set of covariates; as Table 2 indicates, the combined sample size here is “only” about 24,000, and the large-sample approximation is poor. From a business perspective, the treatment intervention in $E_1$ was demonstrably a failure, with an estimated lift that represents a loss of about 0.8%; the treatment in $E_2$ was highly promising — $\hat{\theta} = +6.4\%$ — but (with an outcome variable this noisy) the total sample size of “only” about 257,000 was insufficient to demonstrate its effectiveness convincingly.

Plots such as those in Figure 1 are of extreme interest at company X, because they exhibit the entire y distribution (and its right-tail uncertainties) in a way that sharply underscores the fragility of inferences about means with outcome variables as skewed and kurtotic as those examined here. Moreover, because of the complexity of the distribution involved and the difficulty associated with selecting an appropriate parametric model, it is of key importance to this
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Fig. 2. CJ–BNP posterior distributions (solid curves) for the lift \( \theta \) in \( E_1 \) (upper left) and \( E_2 \) (upper right), with Gaussian approximations (dotted lines) superimposed. Lower left: the \( \theta \) posteriors from \( E_1 \) and \( E_2 \) on the same graph, to give a sense of relative information content in the two experiments; lower right: BNP and approximate-Gaussian posteriors for \( \theta \) in a small subgroup (segment) of \( E_2 \).

Company to understand exactly what is assumed by the statistical model in use, and what is learned from the data. The full logical argument, based on exchangeability and CJ–BNP — and expressed here for the \( T \) group, with an entirely parallel development for \( C \) — is easy to state and understand:

- In the absence of any covariate information about the users in the \( T \) group, Your uncertainty about the real-valued \( y_i^T \) is exchangeable;
- You assume that all statements of uncertainty about propositions \( (A, B, \ldots) \) — of the form \( \mathbb{P}_{CJ}(A \mid B) \) — are real numbers;
- You assume that if You know \( \mathbb{P}_{CJ}(A \mid B) \), then You automatically also know \( \mathbb{P}_{CJ}(\text{not } A \mid B) \);
• You assume that $P_{CJ}(A B | C)$ depends only on $P_{CJ}(A | C)$ and $P_{CJ}(B | AC)$; and

• You assume that tiny changes in Your information can only result in tiny changes in Your probabilities.

Then, by de Finetti’s Representation Theorem for real-valued unknowns, CJ–BNP provides a logically-internally-consistent solution to the problem faced by Company $X$ in a way that

(a) makes full use of all available information judged by You to be relevant, and

(b) does not make use of any information not directly implied by problem context.

Remark 5.6. It appears to us from an extensive literature search that, prior to this work, the theoretical underpinnings of BNP have been quite different to those we present here. The development of BNP in the superb and extraordinarily influential book on Bayesian Theory by Bernardo and Smith (1994), which has been cited almost 5,000 times and which has informed the views of an entire generation of Bayesians, is representative of much previous BNP foundational work, as follows:

• The book begins in the operational-subjective (OS) Bayesian mode of de Finetti and Lad, with a Finitely-Additive probability function $P_{BS}(\cdot)$ developed from natural coherence-based Principles and Axioms aimed at viewing the world decision-theoretically;

• Note, however, that (i) $P_{BS}(A)$ is a function of only one argument and (ii) the input $A$ in $P_{BS}(A)$ is a set, not a proposition (a sharp departure from de Finetti and Cox–Jaynes); in other words, Bernardo and Smith follow Kolmogorov in not taking conditional probability as the primitive concept;

• Bernardo and Smith then use the first 106 pages of their book to explore the implications of their Finitely-Additive probability map, including the useful development of (i) an elegant decision theory and (ii) scoring functions and Kullback-Leibler divergence as quantifications of information;

• However, at this point (on p. 106) they generalize their $P_{BS}(\cdot)$ via a Monotone Continuity Postulate, which immediately implies both Continuity from Above (Kolmogorov’s “Fourth Axiom”) and Countable Additivity (Kolmogorov’s Third Axiom). They then repeat Kolmogorov’s Definition of an $(\Omega, \mathcal{F}, P_{BS})$ probability triple and announce that “From now on, our mathematical development will take place within the assumed structure of a probability [triple],” and for the remaining 478 pages of the book — including a brief supportive section on BNP (on pp. 228–229) — all probability functions are operationally Kolmogorov, not de Finetti or Cox–Jaynes, as conditional probability is defined in the Kolmogorov style: $P_{BS}(A | B) \triangleq \frac{P_{BS}(AB)}{P_{BS}(B)}$ for all $A \in \mathcal{F}$ and for all $B \in \mathcal{F}$ such that $P_{BS}(B) > 0$;

• A final issue must now be addressed: in what sense can Bayesian assertions about parameters be made in the context of a Kolmogorov-style probability function? Bernardo and Smith answer this question by making both data values $D$ and parameter values $\theta$ Kolmogorov-style random variables defined on probability triples, although — in the spirit of de Finetti — such entities are referred to not as random variables but as random quantities. Now posterior distributions such as $P_{BS}(F_T | y^T)$ in the Company–X case study may be properly discussed.

Note by contrast that in the world of Jaynes (2003) there is no foundational appeal to random variables. For example, if $0 < \theta < 1$ is a rate (unknown to You) at which something happens and $D$ is a data set You will observe in the future, to decrease Your uncertainty about $\theta$, define
Table 2. CJ–BNP inferential summaries of lift in the two A/B tests $E_1$ and $E_2$; “full” and “segment” are explained in the text.

| Experiment | Total n  | Posterior for $\theta$ (%) | $\mathbb{P}_{CJ}(0 > 0 | y')$ | BNP   | Gaussian |
|------------|----------|-----------------------------|-------------------------------|-------|----------|
| $E_1$      | 24,465,793 | $-0.818$ 0.608             | 0.0894                        | 0.0892 |
| $E_2$ full | 256,721   | $+6.365$ 14.01             | 0.6955                        | 0.6752 |
| $E_2$ segment | 23,674 | $+5.496$ 34.26             | 0.5075                        | 0.5637 |

$F_0(t) \triangleq \mathbb{P}_{CJ}(\theta \leq t)$, in which $(\theta \leq t)$ is a proposition; as $t$ varies from 0 to 1, the function $F_0(t)$ is your CDF summarizing your (prior) information about $\theta$ external to D. Foundationally, the measure-theoretic random variable notion follows from your description of uncertainty, rather than being used to construct it.

The above Bernardo-Smith theoretical justification of BNP is perfectly sound mathematically, but we find it difficult to explain to newcomers to BNP (such as students and collaborators) who have some understanding of both the frequentist and Bayesian paradigms and to whom the mixture of concepts is mystifying. It seems to us that CJ–BNP, which involves no appeal to non-Bayesian ideas, has substantial interpretive advantages over Bernardo-Smith-BNP; at the very least, it is good to now have both.

Remark 5.7. (Future work) The research agenda summarized here may usefully be extended in at least three ways.

- **(Alternative to Extendability)** Find an axiomatization that can establish the validity of CJ on a lattice (the finite-domain case) without recourse to our Comparative Extendability Axiom.

- **(Quantum probability)** The mathematical foundations of quantum theory and its statistical interpretation require the development of a non-commutative (non-Boolean) and/or non-real-valued version of probability (Youssef (1995)); see, e.g., Goyal et al. (2010), Holik et al. (2014) and Holik et al. (2015), who suggest that the CJ approach may be adapted to this purpose. Our formalism in this paper is ideally suited to this task.

- **(Halpern’s Conjecture)** Given a propositional Boolean algebra $A$, in the language of this paper Halpern (1999a) has conjectured that the class of maps from $[A \times (A \setminus 0)]$ to $[0,1]$ that satisfy only Cox’s original assumptions, without additional Axioms such as Density or Extendability, includes only functions that are “close” (in some unspecified sense) to maps isomorphic to $\mathbb{P}_{CJ}$. A proof of this conjecture would shed useful light on the foundations of probability and statistical theory.

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Appendix

1. Table 3 presents the five Boolean axioms for propositions.

Table 3. The five Boolean axioms.

| (Associativity) | $A \lor (B \lor C) = (A \lor B) \lor C$ | $A \land (B \land C) = (A \land B) \land C$ |
| (Commutativity) | $A \lor B = B \lor A$ | $A \land B = B \land A$ |
| (Identity) | $A \lor 0 = A$ | $A \land 1 = A$ |
| (Distributivity) | $A \lor (B \land C) = (A \lor B) \land (A \lor C)$ | $A \land (B \lor C) = (A \land B) \lor (A \land C)$ |
| (Complementation) | $A \lor (\neg A) = 1$ | $A \land (\neg A) = 0$ |

Table 4. Isomorphism between $(\mathcal{F}, \cap, \cup, \cdot, \emptyset, \Omega)$ and $(\mathcal{A}, \land, \lor, \neg, 0, 1)$ in the die-rolling example, in which $\Omega = \{1, 2, 3, 4, 5, 6\}$; UF = (uppermost face).

| Kolmogorov | Cox–Jaynes |
| $\mathcal{F} = 2^\Omega = \{\emptyset, \{1\}, \ldots, \Omega\}$ | $\mathcal{A} = \{0, (UF = 1), \ldots, 1\}$ |
| $\cap$ (intersection) | $\land$ (and) |
| $\cup$ (union) | $\lor$ (or) |
| $\cdot$ (complement) | $\neg$ (not) |
| $\emptyset$ | 0 = (no roll) |
| $\Omega$ | 1 = (any UF) |

(i) You roll a single die once and record the face that shows uppermost, and You make the assumption that the die-rolling is “fair,” meaning that all six possibilities $\Omega = \{1, 2, 3, 4, 5, 6\}$ have equal probability. Table 4 summarizes the propositional and set-theoretic isomorphism for this problem.

(ii) You draw a real number $y$ from the unit interval $(0, 1)$ and You assume that $(y \mid 1) \sim$ Uniform$(0, 1)$; Table 5 gives the isomorphism.

In both examples, note that the uniformity assumptions apply only to the choices of $P_K$ and $P_{CJ}$; the isomorphism is the same across all $P_K$ and $P_{CJ}$ satisfying their respective sets of Axioms.

References


Table 5. Isomorphism between $(\mathcal{F}, \cap, \cup, \cdot, \emptyset, \Omega)$ and $(\mathcal{A}, \land, \lor, \neg, 0, 1)$ in the uniform draw example, in which $\Omega = (0, 1)$.

| Kolmogorov | Cox–Jaynes |
| $\mathcal{F} =$ the Borel sets on $(0, 1)$ | $\mathcal{A} =$ the smallest Boolean algebra that |
| $\cap$ (intersection) | $\land$ (and) |
| $\cup$ (union) | $\lor$ (or) |
| $\cdot$ (complement) | $\neg$ (not) |
| $\emptyset$ | 0 = (no draw is made) |
| $\Omega$ | 1 = (the draw falls somewhere between 0 and 1) |


